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Geometric Interpretation of Electromagnetism in a Gravitational Theory with Space–Time Torsion

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Abstract

A complete geometric unification of gravity and electromagnetism is proposed by considering two aspects of torsion: its relation to spin established in Einstein–Cartan theory and the possible interpretation of the torsion trace as the electromagnetic potential. Starting with a Lagrangian built of Dirac spinors, orthonormal tetrads, and a complex rather than a real linear connection we define an extended spinor derivative by which we obtain not only a very natural unification, but can also fully clarify the nontrivial underlying fibre bundle structure. Thereby a new type of contact interaction between spinors emerges, which differs from the usual one in Einstein–Cartan theory. The splitting of the linear connection into a metric and an electromagnetic part together with a characteristic length scale in the theory strongly suggest that gravity and electromagnetism have the same geometrical origin.

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1 Introduction

In general relativity the metric $g_{\mu\nu}$ completely determines the linear connection $\Gamma^{\alpha}{}_{\mu\beta}$, which becomes simply the (symmetric) Levi–Civita connection

$$\Gamma^{\alpha}{}_{\mu\beta} = \{^{\alpha}{}_{\mu\beta}\} := \tfrac{1}{2} g^{\alpha\epsilon} \left(\partial_\mu g_{\epsilon\beta} + \partial_\beta g_{\epsilon\mu} - \partial_\epsilon g_{\mu\beta}\right) \quad . \tag{1.1}$$

The space–time geometry is influenced only by mass–energy, which causes curvature via the Einstein equation, and remains unaffected by spin.

In Einstein–Cartan theory (see Ref. [1] and references therein) the connection is only required to be metric, $\nabla_\alpha g_{\mu\nu} = 0$, and is allowed to have nonvanishing torsion $T^{\alpha}{}_{\mu\beta} = \Gamma^{\alpha}{}_{\mu\beta} - \Gamma^{\alpha}{}_{\beta\mu}$, contrary to the torsionless Levi–Civita connection. The structure of the connection now becomes [1]

$$\Gamma^{\alpha}{}_{\mu\beta} = \{^{\alpha}{}_{\mu\beta}\} + \tfrac{1}{2}(T_{\mu}{}^{\alpha}{}_{\beta} + T_{\beta}{}^{\alpha}{}_{\mu} + T^{\alpha}{}_{\mu\beta}) \quad . \tag{1.2}$$

This generalization enables the space–time geometry to respond not only to mass but also to spin, where spinning matter produces torsion. For Dirac particles the torsion is totally antisymmetric in its indices and creates a cubic self-interaction term in the spinor equation [2]

$$i\gamma^\mu \nabla^*_\mu \psi - \frac{mc}{\hbar}\psi + \tfrac{3}{8} l_0^2 (\overline{\psi}\gamma^5\gamma^\delta\psi)\gamma^5\gamma_\delta\psi = 0 \tag{1.3}$$

where ∇^*_μ is the covariant spinor derivative with respect to the Levi–Civita connection, see (3.12), and l_0 is the Planck length. The observed "contact interaction" in (1.3) also contributes to the energy–momentum equation [2].

Besides this well-known aspect of torsion another physical role for it has been suggested in several works on the unification of gravitation and electromagnetism. The idea of such a geometrical unification is to omit any restrictions on the real linear connection $\Gamma^{\alpha}{}_{\mu\beta}$ and to incorporate the electromagnetic phenomena into this extended space–time geometry. More precisely, the electromagnetic vector potential A_μ is identified with the torsion trace $T_\mu = T^{\alpha}{}_{\mu\alpha}$. In the so called nonsymmetric unified field theory, Einstein [3] has considered a general linear connection, but his aim was to incorporate electromagnetism into the metric. He introduced a nonsymmetric metric $\tilde{g}_{\mu\nu} (\neq \tilde{g}_{\nu\mu})$ and identified its antisymmetric part with the

dual of the electromagnetic field strength. His theory was unsuccessful because it could not account for the equation of motion [4,5]. To remedy this and various other shortcomings of Einstein's theory several authors have suggested to make the above mentioned identification $T_\mu \sim A_\mu$ in an ad hoc manner [6,7], still using a nonsymmetric metric. In subsequent works of McKellar [8] and Jakubiec and Kijowski [10] this is achieved without ad hoc assumptions, using the usual symmetric metric. McKellar starts with the metric $g_{\mu\nu}$ and a general linear connection $\Gamma^\alpha{}_{\mu\beta}$ and obtains

$$\Gamma^\alpha{}_{\mu\beta} = \{^\alpha{}_{\mu\beta}\} + \tfrac{1}{3}\delta^\alpha{}_\beta \cdot T_\mu \tag{1.4}$$

as solution of the field equation with the usual convention $\nabla_\mu X^\alpha = \partial_\mu X^\alpha + \Gamma^\alpha{}_{\mu\beta} X^\beta$ for the covariant derivative of a vector field X^α. His field equations taken together resemble the source-free Einstein–Maxwell equations, provided that $T_\mu \sim A_\mu$ holds. In Ref. [9] Ferraris and Kijowski start not with a metric but with $\Gamma^\alpha{}_{\mu\beta}$ alone and arrive at (1.4) but, contrary to McKellar, they regard $\Gamma^\alpha{}_{\mu\alpha}$, which is not a vector, as the electromagnetic potential and deduce a theory of electromagnetism differing from the Maxwell theory, whereas in Ref. [10] Jakubiec and Kijowski return to the identity $T_\mu \sim A_\mu$ and include Dirac spinors in the unification. Unfortunately the employed spinor derivative requires two connections $\Gamma^\alpha{}_{\mu\beta}$ and $\{^\alpha{}_{\mu\beta}\}$ from the very beginning, and furthermore, from the general linear connection $\Gamma^\alpha{}_{\mu\beta}$ only its trace $\Gamma^\alpha{}_{\mu\alpha}$ appears in this derivative. Owing to this insufficient coupling of the linear connection to spinorial matter torsion does not couple to spin and so its important physical role established in Einstein–Cartan theory is missing. Like McKellar, the authors of Ref. [10] do not clarify the fibre bundle structure of their unification; for example, T_μ in (1.4) cannot be gauged with U(1), as there is no U(1)–bundle constructed in the theory. It is thus only a vector but not a potential.

From the discussions of general relativity, Einstein–Cartan theory, and the unified field theories we conclude that a more general linear connection would enable the space–time geometry to incorporate further physical phenomena in addition to gravitation. However, although the GL(4,$\mathbb{R}$)–connection of Refs. [8,9,10] is more general than the metric connection of Einstein–Cartan theory the spin-torsion coupling was missing either because matter was not considered [8,9] or because the

spinor derivative was somewhat inappropriate [10].

In this work our aim is to obtain a new unification of gravity and electromagnetism including the spin-torsion coupling, thus accounting for both aspects of torsion mentioned before. To achieve this we further expand the space-time geometry and allow for complex linear connections. Let us explain why this complex extension is necessary. Obviously, we must introduce a new spinor derivative containing a spin-torsion coupling. As a consequence, we may imagine that matter will be coupled to the connection more tightly and can therefore twist the space–time geometry so strongly that even complex degrees of freedom are excited. Another reason for the complex extension is the fact that in Refs. [8,10] no U(1)–bundle structure could be constructed for the torsion trace T_μ of a real connection.

Using a complex linear connection and an extended spinor derivative we arrive at a new unification of gravity and electromagnetism which fully clarifies the fibre bundle structure, especially the U(1)–bundle. All field equations follow from the variational principle. Due to the special spinor derivative both aspects of torsion in Eqs. (1.3) and (1.4) are slightly altered. The spinor–spinor contact interaction is now found to occur only between distinct particles, thus excluding a self-interaction like in Eq. (1.3).

In Section 2 we establish notation and introduce the Lagrangian density. In Sections 3 and 4 the field equations are derived using the variational method and their physical content is discussed. Here we slightly expand the theory to include three types of charged particles and also consider many particle systems to observe the new type of spinor–spinor contact interaction. In Section 5 a short summary is given. The fibre bundle geometry is discussed in Appendix A.

2 Lagrangian density

The theory rests on the variational principle and employs a Lagrangian density built of orthonormal tetrads, a complex linear connection, and Dirac spinors. To introduce these field variables let us consider a real 4–dimensional space–time manifold M. We assume that M is endowed with a pseudo-riemannian metric $g_{\mu\nu}$ and a space– and time–orientation which follows from the reduction of the frame bundle $F(M)$ to a

special Lorentz bundle $L^+_\uparrow(M)$; this is a principal bundle consisting of orthonormal tangent bases such that the structure group is given by the special orthochronous Lorentz group $L^+_\uparrow$ with Lie algebra $\boldsymbol{l}$

$$\begin{aligned} L^+_\uparrow &:= \Big\{\Lambda \in \mathrm{Mat}(4,\mathbb{R}) | \Lambda^T \eta \Lambda = \eta\,,\ \det\Lambda = 1\,,\ \Lambda^0{}_0 \geq 1\Big\} \ ; \\ \boldsymbol{l} &= \Big\{\Lambda \in \mathrm{Mat}(4,\mathbb{R}) | \Lambda^T \eta + \eta\Lambda = 0\Big\} \ , \end{aligned} \tag{2.1}$$

where $\eta = (\eta_{\mathsf{ab}}) = (\eta^{\mathsf{ab}}) = \mathrm{diag}(1,-1,-1,-1)$. A tetrad $\sigma = (e_{\mathsf{a}}{}^\mu \partial_\mu)$ is a local cross section in $L^+_\uparrow(M)$, where latin indices, running from 0 to 3, are anholonomic and will be lowered and raised with η_{ab} and η^{ab}, respectively. Greek indices run also from 0 to 3 and refer to local coordinates. They are lowered and raised with $g_{\mu\nu}$ and $g^{\mu\nu}$, the latter being the inverse of $g_{\mu\nu}$. Let $(e^{\mathsf{a}}{}_\mu dx^\mu)$ denote the reciprocal tetrad satisfying $e_{\mathsf{a}}{}^\mu e^{\mathsf{a}}{}_\nu = \delta^\mu{}_\nu$ and $e_{\mathsf{a}}{}^\mu e^{\mathsf{b}}{}_\mu = \delta^{\mathsf{b}}{}_{\mathsf{a}}$. We then have the following relations

$$g^{\mu\nu} = e_{\mathsf{a}}{}^\mu e^{\mathsf{a}\nu} \ , \quad g_{\mu\nu} = e_{\mathsf{a}\mu} e^{\mathsf{a}}{}_\nu \ , \quad e := \det(e^{\mathsf{a}}{}_\mu) = \sqrt{-\det(g_{\mu\nu})} \ . \tag{2.2}$$

The components $e_{\mathsf{a}}{}^\mu$ and $e^{\mathsf{a}}{}_\mu$ will be used to convert coordinate indices to anholonomic ones and vice versa.

The complex frame bundle $F_c(M)$ is a GL(4,$\mathbb{C}$)–principal bundle consisting of all complex tangent bases of $\mathbb{C} \otimes TM$. In particular a tetrad σ is a cross section in $F_c(M)$. Therefore a GL(4,$\mathbb{C}$)–connection ω on $F_c(M)$ can be pulled back to M via σ, yielding a $\boldsymbol{gl}(4,\mathbb{C})$–valued 1-form $(\sigma^*\omega)^{\mathsf{a}}{}_{\mathsf{b}} =: \Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}}\, dx^\mu$, which we call a complex linear connection. Its coordinate components $\Gamma^\alpha{}_{\mu\beta} = e_{\mathsf{a}}{}^\alpha\, e^{\mathsf{b}}{}_\beta \Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}} + e_{\mathsf{c}}{}^\alpha \partial_\mu e^{\mathsf{c}}{}_\beta$ transform in the well-known inhomogeneous way under coordinate changes. The curvature tensor, Ricci tensor, curvature scalar, and the curvature trace are defined as follows

$$\begin{aligned} R^{\mathsf{a}}{}_{\mathsf{b}\mu\nu} &= \partial_\mu \Gamma^{\mathsf{a}}{}_{\nu\mathsf{b}} + \Gamma^{\mathsf{a}}{}_{\mu\mathsf{c}} \Gamma^{\mathsf{c}}{}_{\nu\mathsf{b}} - \partial_\nu \Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}} - \Gamma^{\mathsf{a}}{}_{\nu\mathsf{c}} \Gamma^{\mathsf{c}}{}_{\mu\mathsf{b}} \ ; \\ R_{\beta\nu} &= R^{\mathsf{a}}{}_{\mathsf{b}\mu\nu} \cdot e_{\mathsf{a}}{}^\mu\, e^{\mathsf{b}}{}_\beta \ ; \\ R &= R^{\mathsf{a}}{}_{\mathsf{b}\mu\nu} \cdot e_{\mathsf{a}}{}^\mu\, e^{\mathsf{b}\nu} \ ; \\ Y_{\mu\nu} &= R^{\mathsf{a}}{}_{\mathsf{a}\mu\nu} = \partial_\mu \Gamma^{\mathsf{a}}{}_{\nu\mathsf{a}} - \partial_\nu \Gamma^{\mathsf{a}}{}_{\mu\mathsf{a}} \ . \end{aligned} \tag{2.3}$$

Note that $\Gamma^{\mathsf{a}}{}_{\nu\mathsf{a}}$ is a vector, contrary to $\Gamma^\alpha{}_{\nu\alpha}$. This vector vanishes for metric connections on $L^+_\uparrow(M)$ because of the Lie algebra condition $\Gamma_{\mathsf{a}\mu\mathsf{b}} + \Gamma_{\mathsf{b}\mu\mathsf{a}} = 0$.

Every metric connection defines a covariant differentiation of a Dirac spinor [2]

$$\nabla_\mu \psi = \partial_\mu \psi - \tfrac{1}{4}\Gamma_{\mathsf{a}\mu\mathsf{b}}\gamma^{\mathsf{b}}\gamma^{\mathsf{a}}\psi \quad , \tag{2.4}$$

where the γ–matrices satisfy $\gamma^{\mathsf{a}}\gamma^{\mathsf{b}} + \gamma^{\mathsf{b}}\gamma^{\mathsf{a}} = 2\eta^{\mathsf{ab}}\mathbb{1}$, see e. g. Ref. [2]. We extend (2.4) to the case where $\Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}}$ is a complex linear connection. At this stage (2.4) is rather a formal definition as it is only $L^{+}_{\uparrow}$–covariant but not with respect to GL(4,$\mathbb{C}$). The full geometrical meaning of (2.4) is expounded in Appendix A, where we also clarify the nontrivial bundle geometry of our unification scheme.

Introducing $\overline{\psi} := \psi^\dagger\gamma^0$, $\gamma^\mu := \gamma^{\mathsf{a}}\, e_{\mathsf{a}}{}^\mu$, the mass of the spinor particle m, $k = 8\pi G/c^4$, and a length scale l we write down the following Lagrangian density

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_m + \mathcal{L}_G + \mathcal{L}_Y \\ &=: e\cdot\hbar c\left[i\overline{\psi}\gamma^\mu\nabla_\mu\psi - \frac{mc}{\hbar}\overline{\psi}\psi\right] - \frac{e}{2k}R + \frac{e}{4k}l^2 Y_{\mu\nu}Y^{\mu\nu} \quad . \end{aligned} \tag{2.5}$$

Although it is complex valued we do not make it real, because this would restrict the contributions of the full complex connection. For an interesting example of complex Lagrangian theory see e. g. Ref. [11]. Apart from being complex the three parts $\mathcal{L}_m$, $\mathcal{L}_G$, and $\mathcal{L}_Y$ resemble more or less the usual Lagrangian densities of spinorial matter, gravity, and the electromagnetic field, respectively. Whereas expressions similar to $\mathcal{L}_G$ and $\mathcal{L}_Y$ for a real connection were already used in Refs. [8,9,10], the matter Lagrangian $\mathcal{L}_m$ including the extended spinor derivative (2.4) is new and plays a key role in our unification. Note that the partial derivatives ∂_μ and the connection have the dimension of inverse length. Therefore, from purely dimensional arguments, we must introduce a squared length l^2 in $\mathcal{L}_Y$. To compare l with the Planck length $l_0 := \sqrt{\hbar c k}$ we rewrite (2.5) as follows

$$\mathcal{L} = \frac{e}{k}\cdot\left[il_0^2\,\overline{\psi}\gamma^\mu\nabla_\mu\psi - mc^2k\overline{\psi}\psi - \tfrac{1}{2}R + \tfrac{1}{4}l^2 Y_{\mu\nu}Y^{\mu\nu}\right] \quad . \tag{2.6}$$

In the last term we recognize l^2 as the self-coupling constant of the connection, implying that l is an intrinsic length of the space–time geometry. The first term on the right side of (2.6) reveals l_0^2 as the coupling constant between the connection and matter. But if we regard Dirac spinors ultimately as geometrical objects, then l_0 also is a characteristic unit of the space–time. We therefore expect l and l_0 being of the same magnitude.

3 Field equations

The field equations are obtained by varying $\mathcal{L}$ with respect to **(a)** $\Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}}$, **(b)** ψ and $\overline{\psi}$, and **(c)** $e_{\mathsf{a}}{}^{\mu}$. The Euler–Lagrange equation for a representative field variable v is

$$0 = \frac{\delta\mathcal{L}}{\delta v} := \frac{\partial\mathcal{L}}{\partial v} - \partial_\nu \frac{\partial\mathcal{L}}{\partial\partial_\nu v} \quad , \tag{3.1}$$

where higher order derivatives are absent in $\mathcal{L}$. In the following we give only an outline of the computations. For details see Ref. [12].

(a) Define the following complex valued third rank tensor

$$\Sigma^{\alpha}{}_{\mu\beta} := \Gamma^{\alpha}{}_{\mu\beta} - \{^{\alpha}{}_{\mu\beta}\} \ . \tag{3.2}$$

A third rank tensor always admits a "4-vector decomposition"

$$\Sigma_{\alpha\beta\gamma} = Q_\alpha g_{\beta\gamma} + S_\beta g_{\alpha\gamma} + g_{\alpha\beta} U_\gamma - \tfrac{1}{12}\eta_{\alpha\beta\gamma\delta} V^\delta + \Upsilon_{\alpha\beta\gamma} \quad , \tag{3.3}$$

where the four vectors Q_α, S_β, U_γ, V_δ and the "tensor rest" $\Upsilon_{\alpha\beta\gamma}$ are explicitly defined in Appendix B. This tensor rest satisfies $\Upsilon^{\alpha}{}_{\alpha\gamma} = \Upsilon^{\alpha}{}_{\gamma\alpha} = \Upsilon_{\gamma}{}^{\alpha}{}_{\alpha} = \Upsilon_{[\alpha\beta\gamma]} = 0$. We have denoted the volume element by $\eta_{\alpha\beta\gamma\delta} := e \cdot \epsilon_{\alpha\beta\gamma\delta}$, wherein $\epsilon_{\alpha\beta\gamma\delta}$ is totally antisymmetric and $\epsilon_{0123} = 1$. The field equation for $\Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}}$ follows from (2.5) or (2.6), (3.1) and (3.2)

$$\begin{aligned} 0 \ &= \ \frac{\delta\mathcal{L}}{\delta\Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}}} \cdot \delta^{\gamma}{}_{\mu}\, e^{\mathsf{a}\alpha}\, e_{\mathsf{b}}{}^{\beta} \cdot \frac{k}{e} \quad \Leftrightarrow \\ -\tfrac{1}{4} i l_0^2\, \overline{\psi}\gamma^\gamma\gamma^\beta\gamma^\alpha\psi \ &= \ \tfrac{1}{2}\left[\Sigma^{\beta\epsilon}{}_{\epsilon} g^{\alpha\gamma} + \Sigma^{\epsilon}{}_{\epsilon}{}^{\alpha} g^{\gamma\beta} - \Sigma^{\beta\alpha\gamma} - \Sigma^{\gamma\beta\alpha}\right] + l^2 g^{\alpha\beta}\, \nabla^*_\nu Y^{\nu\gamma} \ . \end{aligned} \tag{3.4}$$

Here ∇^*_μ is the covariant differentiation with respect to $\{^{\alpha}{}_{\mu\beta}\}$, see Appendix B. The bracket $[\ldots]$ contains the contributions of $\mathcal{L}_G$, wherein terms belonging to the Levi–Civita connection part in (3.2) and terms created by the last term of (3.1) cancel out completely. We now use (3.3) and the definitions of the vector current $j^\alpha := \overline{\psi}\gamma^\alpha\psi$ and the axial current $j^{5\,\delta} := \overline{\psi}\gamma^5\gamma^\delta\psi$, see Appendix B. Contracting (3.4) successively with $g_{\beta\gamma}$, $g_{\alpha\gamma}$, $g_{\alpha\beta}$ and $1/6 \cdot \eta_{\gamma\beta\alpha\delta}$ we obtain

$$
\begin{array}{ll}
\langle\, g_{\beta\gamma} : \rangle & -il_0^2 \cdot j^\alpha = 3Q^\alpha + 6U^\alpha + l^2 \nabla^*_\nu Y^{\nu\alpha} \\
\langle\, g_{\alpha\gamma} : \rangle & \frac{1}{2} il_0^2 \cdot j^\beta = 6Q^\beta + 3U^\beta + l^2 \nabla^*_\nu Y^{\nu\beta} \\
\langle\, g_{\alpha\beta} : \rangle & -il_0^2 \cdot j^\gamma = 4l^2 \cdot \nabla^*_\nu Y^{\nu\gamma} \\
\langle\, 1/6 \cdot \eta_{\gamma\beta\alpha\delta} : \rangle & -\frac{1}{4} l_0^2 \cdot j^5{}_\delta = -\frac{1}{12} V_\delta
\end{array}
\tag{3.5}
$$

One can easily derive $-Q^\alpha = U^\alpha = -il_0^2/4 \cdot j^\alpha$. Inserting this and the last two equations of (3.5) into (3.4) we obtain $\Upsilon_{\alpha\beta\gamma} = 0$. Since the Levi–Civita connection fulfills $\{_{a\mu b}\} + \{_{b\mu a}\} = 0$, where $\{_{a\mu b}\} = e_{\mathsf{a}\alpha}\, e_{\mathsf{b}}{}^\beta \{^\alpha{}_{\mu\beta}\} + e_{\mathsf{a}\gamma} \partial_\mu e_{\mathsf{b}}{}^\gamma$, it follows $\Gamma^{\mathsf{a}}{}_{\mu\mathsf{a}} = \Sigma^{\mathsf{a}}{}_{\mu\mathsf{a}} = Q_\mu + 4S_\mu + U_\mu = 4S_\mu$ and thus $Y_{\mu\nu} = 4S_{\mu\nu} := 4(\partial_\mu S_\nu - \partial_\nu S_\mu)$. These results amount to

$$
\Gamma^\alpha{}_{\mu\beta} = \widehat{\Gamma}^\alpha{}_{\mu\beta} + \delta^\alpha{}_\beta \cdot S_\mu\ , \quad \text{where} \tag{3.6}
$$

$$
\widehat{\Gamma}^\alpha{}_{\mu\beta} := \{^\alpha{}_{\mu\beta}\} + \frac{1}{4} l_0^2 \left(i \cdot j^\alpha g_{\mu\beta} - i \cdot \delta^\alpha{}_\mu j_\beta - \eta^\alpha{}_{\mu\beta\delta} j^{5\,\delta} \right) \tag{3.7}
$$

and

$$
16il^2/l_0^2\, \nabla^*_\nu S^{\nu\gamma} = j^\gamma \ . \tag{3.8}
$$

The last equation implies the current conservation

$$
\nabla^*_\gamma j^\gamma = 16il^2/l_0^2\, \nabla^*_\gamma \nabla^*_\nu S^{\nu\gamma} = 0 \ . \tag{3.9}
$$

So far we have not used the complex extension of the connection explicitly. But now from (3.7) and (3.8) we see that parts of the connection (3.6) must be complex valued. In other words, these equations can not be solved using a real connection only. This is exactly the reason why we have chosen a complex rather than a real linear connection as our field variable. As shown in Appendix A the connection (3.6) is a sum of the complex metric connection $\widehat{\Gamma}^\alpha{}_{\mu\beta}$ on the complex Lorentz bundle $\mathbb{C}L^+(M)$ and the U(1)–connection S_μ on the trivial U(1)–bundle $M \times \mathrm{U}(1)$, glued together by canonical bundle mappings. If S_μ is a U(1)–connection, it must be purely imaginary, see (3.15). This can not be deduced from (3.8) alone as it contains only $\nabla^*_\nu \mathrm{Re}(S^{\nu\gamma}) = 0$, but not $\mathrm{Re}(S_\mu) = 0$ itself. According to Appendix A the

corresponding U(1)–gauge transformation is given by

$$e_{\mathsf{a}}{}^{\mu} \mapsto e_{\mathsf{a}}{}^{\mu} \ , \quad \tfrac{1}{4}\Gamma^{\mathsf{a}}{}_{\mu\mathsf{a}}(= S_\mu) \mapsto \tfrac{1}{4}\Gamma^{\mathsf{a}}{}_{\mu\mathsf{a}} + \partial_\mu\lambda \ , \quad \psi \mapsto \exp(\lambda)\psi \ . \tag{3.10}$$

(b) The Lagrangian (2.5) immediately yields $0 = \delta\mathcal{L}/\delta\overline{\psi} = \partial\mathcal{L}/\partial\overline{\psi}$ or, equivalently, $i\gamma^\mu\nabla_\mu\psi - mc/\hbar\,\psi = 0$. With (3.6) this can be converted into

$$i\gamma^\mu(\nabla^*_\mu - S_\mu)\psi - \frac{mc}{\hbar}\psi + \tfrac{3}{8}l_0^2(j_\mu + j^5{}_\mu\gamma^5)\gamma^\mu\psi = 0 \ , \tag{3.11}$$

where

$$\nabla^*_\mu\psi := \partial_\mu\psi - \tfrac{1}{4}\{_{a\mu b}\}\gamma^{\mathsf{b}}\gamma^{\mathsf{a}}\psi \tag{3.12}$$

is the covariant spinor differentiation with respect to the Levi–Civita connection. The spinor equation for $\overline{\psi}$ is more difficult to compute [12]. The result is

$$i(\nabla^*_\mu + S_\mu)\overline{\psi}\cdot\gamma^\mu + \frac{mc}{\hbar}\overline{\psi} - \tfrac{3}{8}l_0^2\overline{\psi}(j_\mu + j^5{}_\mu\gamma^5)\gamma^\mu = 0 \tag{3.13}$$

with $\nabla^*_\mu\overline{\psi} = \overline{\nabla^*_\mu\psi}$. The nonlinear terms in (3.11) and (3.13) vanish due to the identity

$$(j_\mu + j^5{}_\mu\gamma^5)\gamma^\mu\psi = 0, \tag{3.14}$$

which can be derived by straightforward but cumbersome computations, recalling $j_\mu = \overline{\psi}\gamma_\mu\psi$ and $j^5{}_\mu = \overline{\psi}\gamma^5\gamma_\mu\psi$ and using e. g. the chiral representation and the properties of the Pauli matrices within the γ–matrices.

Since (3.13) is the spinor equation of the adjoint spinor $\overline{\psi}$, it must agree with the adjoint of the first equation (3.11). This implies that S_μ is purely imaginary,

$$\mathrm{Re}(S_\mu) = 0 \ . \tag{3.15}$$

(c) The Lagrangian (2.5) contains no derivatives of $e_{\mathsf{a}}{}^{\mu}$ and therefore we get

$$\begin{aligned} 0 \ = \ & \frac{\delta\mathcal{L}}{\delta e_{\mathsf{c}}{}^{\alpha}}e_{\mathsf{c}\beta} = \frac{\partial\mathcal{L}}{\partial e_{\mathsf{c}}{}^{\alpha}}e_{\mathsf{c}\beta} \ = \ \left[-\mathcal{L}_m\, g_{\alpha\beta} + e\, i\hbar c\overline{\psi}\gamma_\alpha\nabla_\beta\psi\right] \\ & - \tfrac{1}{2k}\left[-eR\, g_{\alpha\beta} + eR_\alpha{}^{\mu}{}_{\beta\mu} + eR^{\mu}{}_{\alpha\mu\beta}\right] + \tfrac{1}{4k}l^2\left[-eY_{\mu\nu}Y^{\mu\nu}g_{\alpha\beta} + 4eY_{\mu\alpha}Y^{\mu}{}_{\beta}\right] \ . \end{aligned} \tag{3.16}$$

Using (3.3), (3.6) to (3.8) and (3.11) to (3.14) this can be expressed as

$$T^G_{\alpha\beta} = T^m_{\alpha\beta} + T^S_{\alpha\beta} \quad , \tag{3.17}$$

$$T^G_{\alpha\beta} := \tfrac{1}{k}\left(R^*_{\alpha\beta} - \tfrac{1}{2}R^* g_{\alpha\beta}\right) \; ; \tag{3.18}$$

$$T^m_{\alpha\beta} := \tfrac{i\hbar c}{2}\left[\overline{\psi}\gamma_\alpha(\nabla^*_\beta - S_\beta)\psi - (\nabla^*_\beta + S_\beta)\overline{\psi}\cdot\gamma_\alpha\psi + \tfrac{1}{2}\nabla^{*\gamma}(\overline{\psi}\gamma_{[\alpha}\gamma_\beta\gamma_{\gamma]}\psi)\right] \; ; \tag{3.19}$$

$$T^S_{\alpha\beta} := \tfrac{16}{k}l^2\left[S_{\alpha\gamma}S_\beta{}^\gamma - \tfrac{1}{4}S_{\mu\nu}S^{\mu\nu}g_{\alpha\beta}\right] \; . \tag{3.20}$$

In (3.18) $R^*_{\alpha\beta}$ and R^* denote the Ricci–tensor and –scalar for the Levi–Civita connection. Since $T^G_{\alpha\beta}$ and $T^S_{\alpha\beta}$ are symmetric in α and β, (3.17) transfers this property also upon $T^m_{\alpha\beta}$. Indeed, a lengthy calculation [12] gives

$$T^m_{\alpha\beta} = \tfrac{i\hbar c}{4}\left[\overline{\psi}\gamma_\alpha(\nabla^*_\beta - S_\beta)\psi - (\nabla^*_\beta + S_\beta)\overline{\psi}\cdot\gamma_\alpha\psi + (\alpha \leftrightarrow \beta)\right] \quad . \tag{3.21}$$

Since $T^G_{\alpha\beta}$ is proportional to the Einstein–tensor, we obtain

$$0 = \nabla^*_\alpha(T^G)^{\alpha\beta} = \nabla^*_\alpha(T^m)^{\alpha\beta} + \nabla^*_\alpha(T^S)^{\alpha\beta} \quad . \tag{3.22}$$

4 Physical interpretation

We now summarize those field equations which will be discussed in the following

$$16il^2/l_0^2\, \nabla^*_\nu S^{\nu\gamma} = j^\gamma \tag{3.8'}$$

$$e_{\mathsf{a}}{}^\mu \mapsto e_{\mathsf{a}}{}^\mu \;, \quad \tfrac{1}{4}\Gamma^{\mathsf{a}}{}_{\mu\mathsf{a}}(= S_\mu) \mapsto \tfrac{1}{4}\Gamma^{\mathsf{a}}{}_{\mu\mathsf{a}} + \partial_\mu\lambda \;, \quad \psi \mapsto \exp(\lambda)\psi \quad . \tag{3.10'}$$

$$i\gamma^\mu(\nabla^*_\mu - S_\mu)\psi - \frac{mc}{\hbar}\psi = 0 \tag{3.11'}$$

$$i(\nabla^*_\mu + S_\mu)\overline{\psi}\cdot\gamma^\mu + \frac{mc}{\hbar}\overline{\psi} = 0 \tag{3.13'}$$

$$T^G_{\alpha\beta} = T^m_{\alpha\beta} + T^S_{\alpha\beta} \quad . \tag{3.17'}$$

These field equations exhibit precisely the well-known structures of the Einstein–Maxwell theory, provided that S_μ is identified with the electromagnetic potential A_μ

$$S_\mu = \frac{iq}{\hbar c} A_\mu \quad , \tag{4.1}$$

where q is the (positive) elementary charge. In this case, (3.8') is simply the inhomogeneous Maxwell equation, see (4.2), whereas (3.10') describes the electromagnetic U(1)-gauge transformation of a *negatively* charged particle, which we identify with electron. The wave equation (3.11') becomes the corresponding charged spinor equation in a curved space–time, (3.13') being its adjoint. Note that $\nabla^*_\mu - S_\mu$ in (3.11') is the U(1)–gauge covariant spinor derivative. Finally, (3.17') gives the energy–momentum equation involving the energy–momentum tensors of gravity (3.18), charged spinor particle (3.19) or (3.21), and the electromagnetic field (3.20).

In order to fix the length scale l we insert (4.1) into (3.8') and compare it with the usual Maxwell equation

$$\begin{aligned} j^\gamma \stackrel{(4.1)}{=} 16il^2/l_0^2\, \frac{iq}{\hbar c}\, \nabla^*_\nu F^{\nu\gamma} \stackrel{!}{=} \frac{1}{-q}\, \nabla^*_\nu F^{\nu\gamma} \quad \Leftrightarrow \\ l^2 = \frac{1}{64\pi} l_0^2 \frac{\hbar c}{q^2/4\pi} = \frac{1}{64\pi\, \alpha}\, l_0^2 \quad \Rightarrow \quad l \approx 0.83\, l_0 \quad , \end{aligned} \tag{4.2}$$

where α is the fine structure constant and we have employed Heaviside–Lorentz units. As expected in Section 2 the value of l is of the same magnitude as the Planck length, which indicates the close relation of electromagnetism to space–time geometry and to gravity. If we had taken $l := l_0$ in (2.5), we would have obtained $\alpha = 1/64\pi$ and $q \approx 1.32 \cdot 10^{-19}$Coulomb. Renormalization procedures could perhaps improve (4.2) towards $l = l_0$. When (4.1) and (4.2) are taken into account one can easily show that the above mentioned field equations of section 3 are exactly the equations of Einstein–Maxwell theory with an electron. Moreover, with these results the Lagrangian (2.5) can be rewritten as

$$\mathcal{L} = e \cdot \hbar c \left[i\overline{\psi}\gamma^\mu (\nabla^*_\mu - \frac{iq}{\hbar c} A_\mu)\psi - \frac{mc}{\hbar}\overline{\psi}\psi \right] - \frac{e}{2k}\, R^* - \frac{e}{4}\, F_{\mu\nu}F^{\mu\nu} \quad , \tag{4.3}$$

which is the usual Einstein–Maxwell Lagrangian.

Notwithstanding these agreements of our theory with the usual one there are important differences, which will now be discussed. From Appendix A we know

that the "gravitational" metric connection $\widehat{\Gamma}^{\alpha}{}_{\mu\beta}$ (3.6) and the U(1)-potential S_μ emerge out of a single connection by symmetry breaking. In accordance with this geometrical background we regard S_μ as the electromagnetic vector potential rather than A_μ itself. Therefore, we describe the electromagnetic interaction through the field equations in Section 3 together with the definite l (4.2) only, thereby completely disregarding (4.1). The problem of this geometrisation procedure, implying the "melting" of q and A_μ into the single expression S_μ, is, how to incorporate particles with charges different from $-q$. To solve this problem we look closely at the spinor derivative (2.4). Assuming for a moment that $\Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}}$ is metric, $\Gamma_{\mathsf{a}\mu\mathsf{b}} = -\Gamma_{\mathsf{b}\mu\mathsf{a}}$, we can write (2.4) in the following three ways

$$\begin{aligned}
(+) \quad \nabla_\mu\psi &= \partial_\mu\psi - \tfrac{1}{4}\Gamma_{\mathsf{a}\mu\mathsf{b}}\gamma^{\mathsf{b}}\gamma^{\mathsf{a}}\psi = \left(\partial_\mu - \tfrac{1}{8}\Gamma_{\mathsf{a}\mu\mathsf{b}}\left[\gamma^{\mathsf{b}},\gamma^{\mathsf{a}}\right] - \tfrac{1}{4}\Gamma^{\mathsf{a}}{}_{\mu\mathsf{a}}\right)\psi \\
(-) \quad &= \partial_\mu\psi + \tfrac{1}{4}\Gamma_{\mathsf{a}\mu\beta}\gamma^{\mathsf{a}}\gamma^{\mathsf{b}}\psi = \left(\partial_\mu - \tfrac{1}{8}\Gamma_{\mathsf{a}\mu\mathsf{b}}\left[\gamma^{\mathsf{b}},\gamma^{\mathsf{a}}\right] + \tfrac{1}{4}\Gamma^{\mathsf{a}}{}_{\mu\mathsf{a}}\right)\psi \qquad (4.4) \\
(0) \quad &= \left(\partial_\mu - \tfrac{1}{8}\Gamma_{\mathsf{a}\mu\mathsf{b}}\left[\gamma^{\mathsf{b}},\gamma^{\mathsf{a}}\right]\right)\psi \ .
\end{aligned}$$

Turning back to our complex connection the first case (+) is exactly (2.4) and describes a negatively charged particle, whereas (−) and (0) correspond to positively charged and neutral particles with U(1)–gauge transformations $\psi \mapsto \exp(-\lambda)\psi$ and $\psi \mapsto \psi$ under (3.10), respectively.

Consider now a many particle system. We distinguish the particles with index z and classify their charges according to (4.4) by the symbol $\varepsilon = \varepsilon(z)$ taking three values

$$\varepsilon = \varepsilon(z) = +, -, 0. \qquad (4.5)$$

The Lagrangian density of this many particle system reads

$$\begin{aligned}
\widetilde{\mathcal{L}} \;=\; & \sum_z e\hbar c\left[i\overline{\psi}_z\gamma^\mu(\partial_\mu - \tfrac{1}{8}\Gamma_{\mathsf{a}\mu\mathsf{b}}[\gamma^{\mathsf{b}},\gamma^{\mathsf{a}}] - \varepsilon(z)\tfrac{1}{4}\Gamma^{\mathsf{a}}{}_{\mu\mathsf{a}})\psi_z - \frac{m_z c}{\hbar}\overline{\psi}_z\psi_z\right] \\
& -\frac{e}{2k}R + \frac{e}{4k}l^2\,Y_{\mu\nu}Y^{\mu\nu} \ . \qquad (4.6)
\end{aligned}$$

The field equations can be solved in the same manner as in section 3 yielding

$$\Gamma^{\alpha}{}_{\mu\beta} \;=\; \widetilde{\Gamma}^{\alpha}{}_{\mu\beta} + \delta^{\alpha}{}_{\beta}\cdot S_\mu \ , \ \text{ where} \qquad (4.7)$$

$$\widetilde{\Gamma}^{\alpha}{}_{\mu\beta} \;:=\; \{^{\alpha}{}_{\mu\beta}\} + \tfrac{1}{4} l_0^2 \sum_z \left(i \cdot j_z^{\alpha} g_{\mu\beta} - i \cdot \delta^{\alpha}{}_{\mu} j_{z\,\beta} - \eta^{\alpha}{}_{\mu\beta\delta} j_z^{5\,\delta} \right) \tag{4.8}$$

and
$$16i\, l^2/l_0^2\, \nabla^*_{\mu} S^{\mu\nu} = \sum_{\varepsilon(z)=+} j_z^{\nu} - \sum_{\varepsilon(z)=-} j_z^{\nu} \;\;, \tag{4.9}$$

where we have defined $j_z^{\alpha} := \overline{\psi}_z \gamma^{\alpha} \psi_z$ and $j_z^{5\,\delta} := \overline{\psi}_z \gamma^5 \gamma^{\delta} \psi_z$. The energy–momentum equation (3.17) now becomes

$$T^G_{\alpha\beta} = \tilde{T}^m_{\alpha\beta} + T^S_{\alpha\beta} + W_{\alpha\beta} \tag{4.10}$$

with
$$\begin{aligned} \tilde{T}^m_{\alpha\beta} \;=\; & \sum_z e \tfrac{i\hbar c}{4} [\overline{\psi}_z \gamma_{\alpha} (\nabla^*{}_{\beta} - \varepsilon(z) S_{\beta}) \psi_z - (\nabla^*{}_{\beta} + \varepsilon(z) S_{\beta}) \overline{\psi}_z \cdot \gamma^{\alpha} \psi_z \\ & + (\alpha \leftrightarrow \beta)] \;\;, \end{aligned} \tag{4.11}$$

and
$$W_{\alpha\beta} \;=\; e\, \tfrac{3}{8k} l_0^4 \sum_{z \neq z'} \left(j_{z\,\mu}\, j^{\mu}_{z'} + j^{5}_{z\mu}\, j^{5\,\mu}_{z'} \right) g_{\alpha\beta} \;\;. \tag{4.12}$$

The spinor equation (3.11) acquires a new term corresponding to (4.12)

$$i\gamma^{\mu} (\nabla^*_{\mu} - \varepsilon(z) S_{\mu}) \psi_z - \frac{m_z c}{\hbar} \psi_z + \tfrac{3}{8} l_0^2 \sum_{z' \neq z} \left(j^{\mu}_{z'} + j^{5\,\mu}_{z'} \gamma^5 \right) \gamma_{\mu} \psi_z = 0 \;\;. \tag{4.13}$$

We recognize that (4.9) is the correct inhomogeneous Maxwell equation of the many particle system. Eqs. (4.13) are apart from the last contribution the corresponding spinor equations for differently charged particles. Thus, we could treat in a natural way the charges $\pm q$ and 0.

In (4.10) and (4.13) we can observe a spinor–spinor contact interaction between distinct particles, to which both vector and axial currents contribute. The absence of self-interactions among the spinors and also the vanishing of the cubic terms in (3.11) and (3.13) are due to (3.14) and have their origin in our special choice of $\mathcal{L}_m$ in (2.5), where we have omitted the covariant differentiation of $\overline{\psi}$. Usually, the matter Lagrangian is required to be real, neccesitating the inclusion of both covariant derivatives of ψ and $\overline{\psi}$. Since in our case the other Lagrangians $\mathcal{L}_G$ and $\mathcal{L}_Y$ in (2.5) were already complex, there was no need to make $\mathcal{L}_m$ alone real valued via the consideration of the adjoint spinor derivative. In Ref. [2] a real Lagrangian

containing also this covariant derivative of $\overline{\psi}$ led to the self-interaction in (1.2), which is induced only via the axial current $j^{5\mu}$. In our opinion, self-interactions of Dirac particles are unlikely because of the Fermi–Dirac spin-statistics they obey and should be avoided. We remark that spinor–spinor interactions are far too weak to be observed by laboratory experiments, but can influence cosmological and quantum gravitational phenomena [1].

Note that the presence of the vector current j^μ also changes the simple ansatz $T_\mu \sim A_\mu$ for the torsion trace, since from (3.6) we get

$$T_\mu = \Gamma^\alpha{}_{\mu\alpha} - \Gamma^\alpha{}_{\alpha\mu} = 3S_\mu - 3U_\mu = 3\frac{iq}{\hbar c}A_\mu + \tfrac{3}{4} i l_0^2\, j^\mu \quad . \tag{4.14}$$

In addition to these features there is one more aspect of our theory which differs from the usual Einstein–Maxwell theory. A Dirac spinor produces a complex contorsion $1/4l_0^2(i\cdot j^\alpha g_{\mu\beta} - i\cdot\delta^\alpha{}_\mu j_\beta - \eta^\alpha{}_{\mu\beta\delta}j^{5\,\delta})$ in $\widehat{\Gamma}^\alpha{}_{\mu\beta}$ (3.6), which carries out the parallel transports of (uncharged) tensors on M. Since the contorsion is a tensor, it does not vanish even in a local inertial system, which we define to be the special coordinate system around a space–time point $p \in M$ with $g_{\mu\nu}(p) = \mathrm{diag}(1,-1,-1,-1)$ and $\partial_\sigma g_{\mu\nu}(p) = 0$. Although the physical meaning of this phenomenon is not yet clear, we remark that it is invisible to the laboratory experiments of today due to the very small magnitude of l_0^2 in the contorsion. Furthermore, $\widehat{\Gamma}^\alpha{}_{\mu\beta}$ is a metric connection, $\widehat{\nabla}_\mu g_{\alpha\beta} = 0$, where $\widehat{\nabla}_\mu$ is the covariant derivative defined in terms of $\widehat{\Gamma}^\alpha{}_{\mu\beta}$. This guarantees the invariance and conserves the real nature of physical measurements of lengths, time intervals, rest masses and various scalar products of particle momenta.

5 Summary

We have used a complex linear connection and an extended spinor derivative to unify the gravitational and electromagnetic interactions into the space–time geometry. Contrary to other attempts at unification we could also clarify the fibre geometric background (Appendix A). The field equations are derived from a variational principle and exhibit precisely the structures of the Einstein–Maxwell theory with a negatively charged spinor particle. However, the many particle system reveals a new type of spinor–spinor contact interaction, which occurs only between distinct

particles, explaining why it was absent in the single particle case. Furthermore, our theory differs significantly from the usual theory in the geometrical understanding and its physical consequences. From the fibre bundle structure of our unification it follows that a metric connection and an electromagnetic vector potential emerge from the single complex linear connection by symmetry breaking. Accordingly, we interpret electromagnetism purely geometrically and use as the only physical constant a characteristic length l close to the Planck length. This geometrisation scheme involves the "melting" of the charge q and A_μ into one single expression and so rules out the consideration of particles with arbitrary charge. However, spinors with charges $\pm q$ and 0 could be treated in the theory using natural extensions of the covariant spinor differentiation. Finally, the parallel transports of uncharged tensors on space–time are carried out by the resultant metric connection mentioned before, which contains a complex contorsion. Physical measurements remain unaffected by this contribution because the metric is preserved under the parallel displacements.

In our opinion, this complex contorsion gives first hints to a deeper understanding of space–time structure although it is only a very small contribution and unobservable in most cases.

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A Bundle Geometry

We derive the correct geometrical definition of (2.4) and clarify the bundle structure of our unification. In Subsection A.1 we establish the general fibre bundle structure of the theory and especially introduce the extended spin structures essential to the building of the spinor derivative (2.4). In A.2 this fibre geometry is used to obtain a special spin connection ω_s from a given complex linear connection ω.

Considering local cross sections in A.3, we will recognize that ω_s indeed results in the correct spinor derivative (2.4) when written in local form. Here we also observe the symmetry breaking aspect of the connection (3.6). Finally, the properties of the U(1)–gauge transformation are explained in A.4.

For the general theory of fiber bundles see Refs. [13,14], for details concerning the spin geometry we refer to Refs. [15,16] and [12]. In the following all structures are C^∞ or analytic. We denote the Lie group homomorphism and its Lie algebra homomorphism by the same letter.

A.1 Fibre Bundle Structure

Analogously to the real case the complex frame bundle $F_c(M)$ can be reduced to a special complex Lorentz bundle $\mathbb{C}L^+(M)$, the structure group being the special complex Lorentz group $\mathbb{C}L^+$,

$$\mathbb{C}L^+ := \left\{\Lambda \in \mathrm{Mat}(4,\mathbb{C}) | \Lambda^T \eta \Lambda = \eta \, , \, \det \Lambda = 1\right\} \quad ,$$

which is isomorphic to the special orthogonal group SO(4,$\mathbb{C}$). The spin group (more precisely the component topologically connected with the $\mathbb{1}$) corresponding to $\mathbb{C}L^+$ will be denoted by $\mathbb{C}\mathrm{Spin}(\cong \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))$ and the accompanying twofold covering homomorphism by $\xi_o : \mathbb{C}\mathrm{Spin} \to \mathbb{C}L^+$. Using a spin representation $\zeta : \mathbb{C}\mathrm{Spin} \to \mathrm{GL}(4,\mathbb{C})$ we define an enlarged spin group $\mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times$ and the corresponding extended spin representation $\zeta^\times$

$$\zeta^\times : \mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times \to \mathrm{GL}(4,\mathbb{C}) \ , \ \ (A, c) \mapsto \zeta(A) \cdot c^{-1} \quad , \tag{A.1}$$

where $\mathbb{C}^\times$ ($\cong$ GL(1,$\mathbb{C}$)) is the multiplicative group $\mathbb{C}\backslash\{0\}$. The representation c^{-1} was chosen in order to obtain (2.4). Other possible choices c^{+1} and c^0 correspond to $(-)$ and (0) of (4.4), respectively. We further define the homomorphism θ_o by

$$\theta_o : \mathbb{C}L^+ \times \mathbb{C}^\times \to \mathrm{GL}(4,\mathbb{C}) \ , \ \ (\Lambda, c) \mapsto \Lambda \cdot c \quad . \tag{A.2}$$

One can easily show $\mathrm{Ker}(\theta_o) = \{(\mathbb{1}, 1), (-\mathbb{1}, -1)\}$ and that every image Λc has exactly two inverse images (Λ, c) and $(-\Lambda, -c)$. Denote the image group of θ_o by $G := \theta_o(\mathbb{C}L^+ \times \mathbb{C}^\times)$ and its canonical inclusion in GL(4,$\mathbb{C}$) by j_o. Obviously, the

Lie algebra of $\mathbb{C}L^+$ is given by $\mathbb{C} \otimes \boldsymbol{l}$, where $\boldsymbol{l}$ is the Lie algebra of $L^+_\uparrow$ (2.1). The homomorphism θ_o induces an isomorphism of the Lie algebra $\mathbb{C}\otimes\boldsymbol{l} \times \mathbb{C}$ of $\mathbb{C}L^+ \times \mathbb{C}^\times$ onto the Lie algebra $\boldsymbol{g}$ of G, given by $\boldsymbol{g} = \mathbb{C} \otimes \boldsymbol{l} \oplus \mathbb{C} \cdot \mathbb{1}$. We can now write down the following diagram of Lie group homomorphisms

$$\mathrm{GL}(4,\mathbb{C}) \xleftarrow{\zeta^\times} \mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times \xrightarrow{\xi_o \times id} \mathbb{C}L^+ \times \mathbb{C}^\times \xrightarrow{\theta_o} G \xrightarrow{j_o} \mathrm{GL}(4,\mathbb{C}) \tag{A.3}$$

and construct an analogous diagram of principal bundle mappings

$$S^\times(M) \xleftarrow{(*)} (\mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times)(M) \xrightarrow{\xi \times id} (\mathbb{C}L^+ \times \mathbb{C}^\times)(M) \xrightarrow{\theta} G(M) \xrightarrow{j} F_c(M)\ . \tag{A.4}$$

If $G_i(M)$, $i = 1, 2$, are G_i–principal bundles, then $(G_1 \times G_2)(M)$ is the $G_1 \times G_2$–principal bundle given by the restriction of $G_1(M) \times G_2(M)$ to the diagonal $\Delta \subset M \times M$, where Δ is identified with M itself [13]. $\mathbb{C}^\times(M)$ is the trivial $\mathbb{C}^\times$–bundle $M \times \mathbb{C}^\times$, and $\mathbb{C}\mathrm{Spin}(M)$ is a $\mathbb{C}\mathrm{Spin}$–bundle with the corresponding spin structure $\xi : \mathbb{C}\mathrm{Spin}(M) \to \mathbb{C}L^+(M)$ satisfying $\xi(uA) = \xi(u)\xi_o(A)$ for $u \in \mathbb{C}\mathrm{Spin}(M)$ and $A \in \mathbb{C}\mathrm{Spin}$. $S^\times(M)$ is the extended spinor bundle defined to be the associated vector bundle $S^\times(M) = (\mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times)(M) \times_{\zeta^\times} \mathbb{C}^4$, $(*)$ denoting this building procedure. Note that an element ϕ of $S^\times(M)$ is an equivalence class [13]

$$\phi = [u, \phi_o] = [u \cdot (A, c)\, ,\ \zeta^\times(A, c)^{-1} \cdot \phi_o]\ \ , \tag{A.5}$$

where $u \in (\mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times)(M)$, $\phi_o \in \mathbb{C}^4$ and $(A, c) \in \mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times$. $G(M)$ is the G–principal bundle consisting of elements $(c \cdot X^a)$, where $c \in \mathbb{C}^\times$ and $(X^a) \in \mathbb{C}L^+(M)$. It is thus contained in $F_c(M)$, j denoting the canonical inclusion. Finally, θ is defined as follows

$$\theta : (\mathbb{C}L^+ \times \mathbb{C}^\times)(M) \to G(M)\ ,\ \ ((X^a), c) \mapsto (c \cdot X^a)\ \ . \tag{A.6}$$

We remark that the bundle mappings $\xi \times id$, θ, and j have as their corresponding Lie group homomorphisms exactly $\xi_o \times id$, θ_o, and j_o.

A.2 Spin Connection

Having explained (A.4) we now construct the spinor derivative (2.4) from a single complex linear connection ω on $F_c(M)$. First define the vector subspace $\boldsymbol{m}$ in the

Lie algebra $\boldsymbol{gl}(4,\mathbb{C})$ of $\mathrm{GL}(4,\mathbb{C})$

$$\boldsymbol{m} := \left\{ A \in \boldsymbol{gl}(4,\mathbb{C}) | A^T\eta - \eta A = 0 \text{ and } \mathrm{Tr}(A) = 0 \right\} \quad . \tag{A.7}$$

Then one can easily verify the following (vector space) decomposition

$$\begin{array}{cccccccc} \boldsymbol{gl}(4,\mathbb{C}) & = & \mathbb{C}\otimes \boldsymbol{l} & \oplus & \mathbb{C}\cdot \mathbb{1} & \oplus & \boldsymbol{m} \\ A & = & \frac{1}{2}(A-\eta A^T\eta) & + & \frac{1}{4}\mathrm{Tr}A\cdot\mathbb{1} & + & \frac{1}{2}(A+\eta A^T\eta - \frac{1}{2}\mathrm{Tr}A\cdot\mathbb{1}) \ . \end{array} \tag{A.8}$$

Now, $\boldsymbol{g} = \mathbb{C}\otimes \boldsymbol{l} \oplus \mathbb{C}\cdot\mathbb{1}$ is the Lie algebra of G, and for every $\Lambda c \in G$ it follows $(\Lambda c)\boldsymbol{m}(\Lambda c)^{-1} \subset \boldsymbol{m}$. In this case the $\boldsymbol{g}$–component of $\omega|_{G(M)}$ is a G–connection on $G(M)$ (see Ref. [13]), which we denote by ω_G. Using the Lie algebra isomorphism θ_o we can convert the pull–back $\theta^*\omega_G$ into a connection $\theta_o^{-1}\theta^*\omega_G$ on $(\mathbb{C}L^+\times\mathbb{C}^\times)(M)$. By the same token $\omega_s := (\xi_o\times id)^{-1}(\xi\times id)^*(\theta_o^{-1}\theta^*\omega_G)$ is a connection on $(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M)$, which we call the extended spin connection.

Using the canonical bundle projections $f_{cl} : (\mathbb{C}L^+\times\mathbb{C}^\times)(M) \to \mathbb{C}L^+(M)$ and $f_{c\times} :\to \mathbb{C}^\times(M)$ together with the concomitant group projections $f_{o\,cl} : \mathbb{C}L^+\times\mathbb{C}^\times \to \mathbb{C}L^+$ and $f_{o\,c\times} :\to \mathbb{C}^\times$ we get the connections $\omega_{cl} := f_{o\,cl}(\theta_o^{-1}\theta^*\omega_G)|_{\mathbb{C}L^+(M)}$ and $\omega_{c\times} := f_{o\,c\times}(\theta_o^{-1}\theta^*\omega_G)|_{\mathbb{C}^\times(M)}$, which can be used to express $\theta_o^{-1}\theta^*\omega_G$ in accordance with the fiber product structure of the underlying bundle

$$\theta_o^{-1}\theta^*\omega_G = f_{cl}^*\omega_{cl} \oplus f_{c\times}^*\omega_{c\times} \quad . \tag{A.9}$$

This leads to a corresponding decomposition of the extended spin connection, where we omit the bundle projections for the sake of simplicity

$$\omega_s = \xi_o^{-1}\xi^*\omega_{cl} \oplus \omega_{c\times} \quad . \tag{A.10}$$

A.3 Spinor Derivative

This spin connection ω_s defines a covariant differentiation on the associated vector bundle $S^\times(M)$. We now show that (2.4) is exactly this derivative. Let $\mathcal{U} \subset M$ be an open set and introduce cross sections $\hat{\sigma}$ in $\mathbb{C}\mathrm{Spin}(M)|_{\mathcal{U}}$ and $\hat{1}$ in $\mathbb{C}^\times(M)|_{\mathcal{U}}$, where $\hat{1}$ prescribes to each $p \in \mathcal{U}$ the value $\hat{1}(p) := (p,1) \in \mathbb{C}^\times(M)$. $\xi(\hat{\sigma})$ is a cross

section in $\mathbb{C}L^+(M)$, that is, a complex tetrad. Although we have used in our theory only real tetrads, the following considerations remain valid even if $\xi(\hat\sigma)$ is complex valued. Let $\Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}}\, dx^\mu := (\xi\,(\hat\sigma)\,{}^*\omega)\,{}^{\mathsf{a}}{}_{\mathsf{b}}$ be the $\boldsymbol{gl}(4,\mathbb{C})$–components of the pulled–back connection. Using (A.8) we obtain

$$(\xi(\hat\sigma)^*\omega_{cl})\,{}^{\mathsf{a}}{}_{\mathsf{b}} \;=\; \tfrac{1}{2}(\Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}} - \Gamma_{\mathsf{b}\mu}{}^{\mathsf{a}}) \qquad \text{and} \tag{A.11}$$

$$\hat 1^*\omega_{c\times} \;=\; \tfrac{1}{4}\Gamma^{\mathsf{c}}{}_{\mu\mathsf{c}} \;\; . \tag{A.12}$$

For the whole connection $\theta_o^{-1}\theta^*\omega_G$ the decomposition (A.9) implies

$$\Big((\xi(\hat\sigma), \hat 1)^*\theta_o^{-1}\theta^*\omega_G\Big)\,{}^{\mathsf{a}}{}_{\mathsf{b}} = \tfrac{1}{2}(\Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}} - \Gamma_{\mathsf{b}\mu}{}^{\mathsf{a}}) + \tfrac{1}{4}\Gamma^{\mathsf{c}}{}_{\mu\mathsf{c}}\cdot\delta^{\mathsf{a}}{}_{\mathsf{b}} \;\; . \tag{A.13}$$

The solution (3.6) has exactly this structure and can thus be understood as a sum of two connections (A.11) and (A.12) on two different bundles according to (A.9).

A Dirac spinor ψ is a cross section in $S^\times(M)$. Since $(\hat\sigma, \hat 1)$ is an element of $(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M)$, we can write ψ as an equivalence class according to (A.5)

$$\psi = \Big[(\hat\sigma, \hat 1)\,,\, \psi_{(\hat\sigma,\hat 1)}\Big] \;\;, \tag{A.14}$$

where $\psi_{(\hat\sigma,\hat 1)}$ is a $\mathbb{C}^4$–valued function on $\mathcal{U}$, usually denoted simply by the same letter ψ and referred to as the Dirac spinor itself. This convention was already used in (2.4). The local trivialization of $S^\times(M)$ in (A.14) allows us to express the covariant derivative of ψ through [14]

$$\nabla_\mu\psi = \Big[(\hat\sigma, \hat 1)\,,\, \partial_\mu\psi_{(\hat\sigma,\hat 1)} + \zeta^\times\Big((\hat\sigma, \hat 1)^*\omega_s\Big)\cdot\psi_{(\hat\sigma,\hat 1)}\Big] \;\;, \tag{A.15}$$

where

$$\begin{aligned}
\zeta^\times\Big((\hat\sigma, \hat 1)^*\omega_s\Big) &\overset{(A.10)}{=} \zeta\circ\xi_o^{-1}\xi(\hat\sigma)^*\omega_{cl}\,\oplus\,(-\mathbb{1})\cdot\hat 1^*\omega_{c\times} \\
&= \tfrac{-1}{4}\gamma^{\mathsf{b}}\gamma_{\mathsf{a}}\cdot\tfrac{1}{2}(\Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}} - \Gamma_{\mathsf{b}\mu}{}^{\mathsf{a}}) \;-\; \tfrac{1}{4}\Gamma^{\mathsf{c}}{}_{\mu\mathsf{c}}\cdot\mathbb{1} \\
&= -\tfrac{1}{4}\Gamma^{\mathsf{a}}{}_{\mu\mathsf{b}}\gamma^{\mathsf{b}}\gamma_{\mathsf{a}} \;\; .
\end{aligned} \tag{A.16}$$

This shows the required agreement with (2.4). Note that we have used here the (usual) explicit form of the Lie algebra homomorphism $\zeta\xi_o^{-1}$, see [15,12]. The $-\mathbb{1}$ in front of $\hat 1^*\omega_{c\times}$ is due to the special choice c^{-1} in the representation (A.1), whereas the important factor 1/4 comes from the decomposition (A.8).

A.4 U(1)–Gauge Transformation

If we change the section $\hat{1}$ to $\exp(\lambda)\cdot\hat{1}$, λ being a $\mathbb{C}$-valued function, then from standard theories on gauge transformations it follows

$$\Big(\exp(\lambda)\cdot\hat{1}\Big)^{*}\omega_{c\times}=\tfrac{1}{4}\Gamma^{\mathsf{c}}{}_{\mu\mathsf{c}}+\partial_\mu\lambda$$

and
$$\psi=\Big[(\hat{\sigma},\hat{1})\,,\,\psi_{(\hat{\sigma},\hat{1})}\Big]\overset{(A.1)}{=}\Big[(\hat{\sigma},\exp(\lambda)\cdot\hat{1})\,,\,\exp(\lambda)\cdot\psi_{(\hat{\sigma},\hat{1})}\Big]\quad. \tag{A.17}$$

Since this $\mathbb{C}^{\times}$–gauge transformation takes place on $\mathbb{C}^{\times}(M)$ the tetrad $\xi(\hat{\sigma})$ on $\mathbb{C}L^{+}(M)$ remains unchanged. It is easy to show that the adjoint spinor $\overline{\psi}$ transforms to $\exp(\overline{\lambda})\cdot\overline{\psi}$, and that in (2.5) only the matter Lagrangian $\mathcal{L}_m$ is affected by the change and transforms to $\exp(\lambda+\overline{\lambda})\cdot\mathcal{L}_m$. The invariance of $\mathcal{L}_m$ thus implies $\lambda+\overline{\lambda}=0$ or $\exp(\lambda)\in\mathrm{U}(1)$. We must therefore replace $\mathbb{C}^{\times}(M)$ by its reduced bundle $\mathrm{U}(1)(M):=M\times\mathrm{U}(1)$.

B 4-Vector Decomposition

Given a third rank tensor $\Sigma_{\alpha\beta\gamma}$ define $\Upsilon_{\alpha\beta\gamma}$, Q_α, S_β, U_γ, and V_δ as follows:

$$\begin{aligned}\Sigma_{\alpha\beta\gamma}=:\ &\tfrac{1}{18}\Big[\ \Big(5\,\Sigma_{\alpha}{}^{\epsilon}{}_{\epsilon}-\Sigma^{\epsilon}{}_{\alpha\epsilon}-\Sigma^{\epsilon}{}_{\epsilon\alpha}\Big)\,g_{\beta\gamma}\\ &+\Big(-\Sigma_{\beta}{}^{\epsilon}{}_{\epsilon}+5\,\Sigma^{\epsilon}{}_{\beta\epsilon}-\Sigma^{\epsilon}{}_{\epsilon\beta}\Big)\,g_{\alpha\gamma} &\text{(B.1)}\\ &+\Big(-\Sigma_{\gamma}{}^{\epsilon}{}_{\epsilon}-\Sigma^{\epsilon}{}_{\gamma\epsilon}+5\,\Sigma^{\epsilon}{}_{\epsilon\gamma}\Big)\,g_{\alpha\beta}\Big]+\Sigma_{[\alpha\beta\gamma]}+\Upsilon_{\alpha\beta\gamma}\\ =:\ &\quad Q_\alpha\,g_{\beta\gamma}+S_\beta\,g_{\alpha\gamma}+U_\gamma\,g_{\alpha\beta}-\tfrac{1}{12}\eta_{\alpha\beta\gamma\delta}V^{\delta}+\Upsilon_{\alpha\beta\gamma}\quad. &\text{(B.2)}\end{aligned}$$

For V^{δ} we have then $V_\delta=2\eta_{\alpha\beta\gamma\delta}\cdot\Sigma^{\alpha\beta\gamma}$. From (B.1) we conclude

$$
\begin{aligned}
\Sigma_{\alpha}{}^{\epsilon}{}_{\epsilon} = \tfrac{1}{18}\Big[\ & 20\,\Sigma_{\alpha}{}^{\epsilon}{}_{\epsilon} - 4\,\Sigma^{\epsilon}{}_{\alpha\epsilon} - 4\,\Sigma^{\epsilon}{}_{\epsilon\alpha} \\
& - \Sigma_{\alpha}{}^{\epsilon}{}_{\epsilon} + 5\,\Sigma^{\epsilon}{}_{\alpha\epsilon} - \Sigma^{\epsilon}{}_{\epsilon\alpha} \\
& - \Sigma_{\alpha}{}^{\epsilon}{}_{\epsilon} - \Sigma^{\epsilon}{}_{\alpha\epsilon} + 5\,\Sigma^{\epsilon}{}_{\epsilon\alpha}\Big] + \Upsilon_{\alpha}{}^{\epsilon}{}_{\epsilon} \quad \Leftrightarrow \quad \Upsilon_{\alpha}{}^{\epsilon}{}_{\epsilon} = 0 \ .
\end{aligned}
\tag{B.3}
$$

Similarly, $\Upsilon^{\epsilon}{}_{\beta\epsilon} = \Upsilon^{\epsilon}{}_{\epsilon\gamma} = 0$. Further observe:

$$
\Sigma_{[\alpha\beta\gamma]} = \Sigma_{[\alpha\beta\gamma]} + \Upsilon_{[\alpha\beta\gamma]} \Leftrightarrow \Upsilon_{[\alpha\beta\gamma]} = 0 \ . \tag{B.4}
$$

For the last term in (3.4) note $Y^{\nu\gamma} = -Y^{\gamma\nu}$ and $\{^{\nu}{}_{\nu\mu}\} = \partial_{\mu} e$, so that

$$
\nabla^{*}_{\nu} Y^{\nu\gamma} = \partial_{\nu} Y^{\nu\gamma} + \{^{\nu}{}_{\nu\mu}\} Y^{\mu\gamma} + \{^{\gamma}{}_{\nu\mu}\} Y^{\nu\mu} = \frac{1}{e}\partial_{\nu}(e Y^{\nu\gamma}) \ . \tag{B.5}
$$

For the axial current observe $1/6\,\eta_{\gamma\beta\alpha\delta}\gamma^{\gamma}\gamma^{\beta}\gamma^{\alpha} = i\gamma^{5}\gamma_{\delta}$, where $\gamma^{5} = i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}$, see e. g. [2].

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